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LIOUVILLE BOOTSTRAP VIA HARMONIC ANALYSIS ON A NONCOMPACT QUANTUM GROUP

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ABSTRACT. The purpose of this short note is to announce results that amount to a verification of the bootstrap for Liouville theory in the generic case under certain assumptions concerning existence and properties of fusion transformations. Under these assumptions one may characterize the fusion and braiding coefficients as solutions of a system of functional equations that follows from the combination of consistency requirements and known results. This system of equations has a unique solution for irrational central charge $c > 25$. The solution is constructed by solving the Clebsch-Gordan problem for a certain continuous series of quantum group representations and constructing the associated Racah-coefficients. This gives an explicit expression for the fusion coefficients. Moreover, the expressions can be continued into the strong coupling region $1 < c < 25$, providing a solution of the bootstrap also for this region.

1. INTRODUCTION

Liouville theory or close relatives of it such as the H_3^+ or $SL(2)/U(1)$ WZNW models play a central role in a variety of string-theoretical or gravitational models. These models are simple enough to justify the hope for exact results yet rich enough to capture some important aspects of the physics of strings on nontrivial (maybe curved) backgrounds. From another point of view, these are the natural starting points for beginning to investigate *noncompact* conformal field theories (CFT), i.e. CFT with a continuous spectrum of primary fields. This has motivated a lot of effort towards the exact solution of these models.

Both from the point of view of applications and of the intrinsic structure of the CFT one may consider the determination of the spectrum Virasoro-representations and of the three point function to be central objectives. Concerning the former, a reasonable conjecture was obtained in the early work [1] (stated in (1) below). As far as the three point function is concerned, important progress was initiated by the papers [3, 4] where an explicit formula was proposed and checked in various ways. A method to derive this formula from conditions of consistency of the bootstrap with a spectrum as proposed in [1] was subsequently given in [6]. Further confirmation from a path-integral point of view was more recently given in [7].

Given knowledge of conformal symmetry, spectrum and three point functions one has in principle an unambigous construction for any genus zero correlation function by summing over intermediate states. But the decomposition of a n-point function as sum over three point functions can in general be performed in different ways. Equality of the expressions resulting from different such decompositions ($\leftrightarrow$ locality, crossing symmetry) can be seen as being *the* most difficult sufficient condition to verify for showing consistency of the CFT as characterized by spectrum and three point functions.

The present note will outline an approach to verify the consistency of the bootstrap with spectrum and three point functions as proposed in [1, 3, 4]. The main assumption underlying our approach is the existence of duality transformations for generic conformal blocks that are consistent with (at least part of) the spectrum being continuous. It is explained that the coefficients that describe these transformations will then be severely constrained by consistency conditions of Moore-Seiberg [8][9] type. In fact, taking into account known results on fusion of degenerate Virasoro-representations with generic ones one may derive a system of functional equations for the fusion coefficients. For irrational central charge $c > 25$ it is possible to show that this system of functional equations has at most one solution.

In order to solve these conditions, the ansatz is made that the fusion coefficients should be essentially given as Racah-coefficients for an appropriate continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, with deformation parameter $q = \exp(\pi i b^2)$, related to the central charge $c > 25$ of the Virasoro algebra via $c = 1 + 6(b + b^{-1})^2$. This is motivated by the long history of research on connections between Liouville theory and quantum groups going back to [10, 11, 12] and more recently in particular [13, 14, 15]. The set of quantum group representations considered here is unusual, however: It is neither of lowest nor highest weight, but rather similar to the principal continuous series of representations of $SL(2,\mathbb{R})$. Racah-coefficients are constructed from the solution of the Clebsch-Gordan problem for these representations, explicitly calculated and shown to provide a solution of all the conditions from Liouville theory discussed previously. In particular, the Clebsch-Gordan calculus yields orthogonality relations for the Racah-coefficients which are just what is needed to establish crossing symmetry of general four point functions constructed in terms of the spectrum and the three point functions proposed in [1, 3, 4].

It is worth noting that the quantum group we are discussing has no proper classical counterpart, which is related to a remarkable self-duality under replacing the deformation parameter $q = \exp(\pi i b^2)$ by $\tilde{q} = \exp(\pi i b^{-2})$. This symmetry directly corresponds to the symmetry of Liouville theory under $b \to b^{-1}$ which is natural from the point of view of the bootstrap[1] and encoded in the three point functions of [3, 4]. [2] Moreover, this duality ensures that all the relevant properties of the fusion coefficients will remain true when continuing to $|b| = 1$, corresponding to the strong coupling region $1 < c < 25$. The present work thus also verifies the conjecture of [4] that the bootstrap remains consistent for $1 < c < 25$ when using the obvious continuations of the spectrum and the three point functions that were proposed in [1, 3, 4].

Most of our results are only announced in the present note. More details and rigorous proofs of our quantum group results will appear in a series of publications in preparation [16, 17, 18].

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2. Bootstrap for Liouville theory

The possibility to preserve conformal invariance as a symmetry in the quantization of Liouville theory [1][19] suggests to use the bootstrap formalism [20] as refined in [8][9] in order to exploit the information about correlation functions that is provided by this symmetry most efficiently.

Conformal invariance requires that the Hilbert-space $\mathcal{H}$ decomposes as direct sum (or integral) over tensor products $\mathcal{V}_\alpha \otimes \mathcal{V}_{\tilde{\alpha}}$ of highest weight representations of the left/right Virasoro algebras. The label α is related to the highest weight $h(\alpha)$ of the representation $\mathcal{V}_\alpha$ via $h(\alpha) = \alpha(Q - \alpha)$

[1] The conformal blocks depend on b only via $Q = b + b^{-1}$

[2] The importance of this self-duality for Liouville theory was also observed by L. Faddeev quite a while ago from a rather different point of view.

where $Q = b + b^{-1}$ is related to the central charge c of the Virasoro algebra via $c = 1 + 6Q^2$. Arguments based on canonical quantization suggest [1] the following spectrum for Liouville theory:

$$\mathcal{H} = \int_{\mathbb{S}}^{\oplus} d\alpha \; \mathcal{V}_\alpha \otimes \mathcal{V}_\alpha. \qquad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+ \tag{1}$$

Accordingly the spectrum is expected to be simple, purely continuous and diagonal.

The main object of interest are correlation functions of Virasoro primary fields $V_\alpha(z)$, $z \in \mathbb{C}$ with conformal dimension $h(\alpha)$. Only the genus zero case will be considered in the present paper. It should be possible to evaluate any correlation function such as $\langle 0|\prod_{i=1}^N V_{\alpha_i}(z_i)|0\rangle$ by summing over intermediate states, leading to a representation of that correlation function in terms of the matrix elements $\langle \alpha_3, d_3|V_{\alpha_2}(z)|\alpha_1, d_1\rangle$. The sum over intermediate states splits into integrations over the intermediate representations and summations over vectors within fixed intermediate representations $\mathcal{V}_\alpha \otimes \mathcal{V}_\alpha$. The contributions for fixed intermediate representations turn out to be uniquely given by conformal symmetry in terms of the matrix elements $C(Q-\alpha_3, \alpha_2, \alpha_1) \equiv \langle \alpha_3|V_{\alpha_2}(1)|\alpha_1\rangle$ between highest weight states $\langle \alpha_3|$ and $|\alpha_1\rangle$. In the example of the four point function $\langle 0|\prod_{i=1}^4 V_{\alpha_i}(z_i)|0\rangle$ one thereby arrives at a representation of the form

$$\begin{aligned} &\langle 0|V_{\alpha_4}(z_4)\ldots V_{\alpha_1}(z_1)|0\rangle = \\ &\quad = \int\limits_{\mathbb{S}_{43|21}} d\alpha_{21}\; C(\alpha_4, \alpha_3, \alpha_{21}) C(\bar{\alpha}_{21}, \alpha_2, \alpha_1) \left|\mathcal{F}^s_{\alpha_{21}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{z})\right|^2, \end{aligned} \tag{2}$$

where $\bar{\alpha} \equiv Q - \alpha$ so that $\bar{\alpha} = \alpha^*$ iff $\alpha \in \mathbb{S}$. The *conformal blocks* $\mathcal{F}^s_{\alpha_{21}}$ are represented by power series of the form

$$\begin{aligned} \mathcal{F}^s_{\alpha_{21}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{z}) = z_{43}^{h_2+h_1-h_4-h_3}\, z_{42}^{-2h_2}\, z_{41}^{h_3+h_2-h_4-h_1}\, z_{31}^{h_4-h_1-h_2-h_3}\, \cdot \\ \cdot\, z^{h(\alpha_{21})-h_2-h_1} \sum_{n=0}^{\infty} z^n \mathcal{F}^s_{\alpha_{21},n}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right], \end{aligned} \tag{3}$$

where $h_i = h(\alpha_i)$, $z_{ji} = z_j - z_i$, $j, i = 1, \ldots, 4$ and $z = \frac{z_{43}z_{21}}{z_{42}z_{31}}$. Conformal symmetry uniquely determines the coefficients $\mathcal{F}^s_{\alpha_{21},n}$, $n > 0$ in terms of $\mathcal{F}^s_{\alpha_{21},0}$ which will be chosen as unity.

Remark 1. One should note that the set $\mathbb{S}_{43|21}$ that appears in (2) will in general *not coincide* with the spectrum [21]. This will only be the case when the states $V_{\alpha_2}(z_2)V_{\alpha_1}(z_1)|0\rangle$ and $\langle 0|V_{\alpha_4}(z_4)V_{\alpha_3}(z_3)$ (suitably smeared over $z_4, \ldots, z_1$) are normalizable, which one indeed expects [21][3] to be the case if the parameters $\alpha_4, \ldots, \alpha_1$ satisfy

$$\begin{aligned} 2|\mathrm{Re}(\alpha_1 + \alpha_2 - Q)| < Q \qquad 2|\mathrm{Re}(\alpha_1 - \alpha_2)| < Q \\ 2|\mathrm{Re}(\alpha_3 + \alpha_4 - Q)| < Q \qquad 2|\mathrm{Re}(\alpha_3 - \alpha_4)| < Q. \end{aligned} \tag{4}$$

Otherwise one can have contributions from intermediate representations that do not belong to $\mathbb{S}$, but are well-defined and uniquely determined by considering the analytic continuation of expression (2) from (4) to generic complex $\alpha_4, \ldots, \alpha_1$, as was explained in the example of the H_3^+-WZNW model in [5]. It will therefore not be a loss of generality to restrict attention to the range (4) where one indeed has $\mathbb{S}_{43|21} = \mathbb{S}$.

[3] This can be alternatively found by considering asymptotics of wave-functions in some refined version of the canonical quantization of [1], as will be explained in more detail elsewhere

The second basic property that the operators $V_\alpha(z)$ are required to satisfy is mutual locality $[V_\alpha(z), V_\beta(w)] = 0$ for $z \neq w$. It follows that the four point function can alternatively be represented e.g. as

$$
\langle 0|V_{\alpha_4}(z_4)\dots V_{\alpha_1}(z_1)|0\rangle = \tag{5}
$$
$$
= \int_{\mathbb{S}} d\alpha_{32}\; C(\alpha_4,\alpha_{32},\alpha_1)C(\bar{\alpha}_{32},\alpha_3,\alpha_2)\;\left|\mathcal{F}^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z})\right|^2,
$$

where the *t-channel*[4] conformal blocks are given by power series similar to (3) with z replaced by $1-z$.

Equality of expressions (2) and (5) can be considered as an infinite system of equations for the data $C(\alpha_3,\alpha_2,\alpha_1)$ and $\mathcal{S}$ with coefficients $\mathcal{F}^s_{\alpha_{21}}, \mathcal{F}^t_{\alpha_{32}}$ given by conformal symmetry. As means for the practical determination of these data it is useless, though.

It turns out, however, that the explicit form of the $C(\alpha_3,\alpha_2,\alpha_1)$ can be determined by considering certain special cases of these conditions where the conformal blocks are known explicitly [6]. Under certain assumptions one finds the formula previously proposed in [3, 4] as unique solution:

$$
C(\alpha_3,\alpha_2,\alpha_1) = \left[\pi\mu\frac{\Gamma(b^2)}{\Gamma(1-b^2)}b^{2-2b^2}\right]^{b^{-1}(Q-\alpha_1-\alpha_2-\alpha_3)}\cdot \tag{6}
$$
$$
\cdot\frac{\Upsilon_0\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_3-\alpha_2)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)},
$$

where a definition of the function $\Upsilon_b(x)$ can be found in the Appendix.

The main problem that needs to be solved in order to put the bootstrap onto firmer ground is the verification that locality (or crossing symmetry) is indeed satisfied when $C(\alpha_3,\alpha_2,\alpha_1)$ as given in (6) and $\mathbb{S}$ as given in (1) are used in the construction of correlation functions. At this point we need to introduce our fundamental assumption:

Conjecture 1. There exist invertible fusion-transformations between s- and t-channel conformal blocks:

$$
\mathcal{F}^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z}) = \int_{\mathbb{S}} d\alpha_{32}\; F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\mathcal{F}^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z}). \tag{7}
$$

This conjecture is supported by an explicit calculation in a case with a special choice of $\alpha_4,\dots,\alpha_1$, but *arbitrary* α_{21} where an explicit expression is available thanks to the work of A. Neveu (unpublished).

Remark 2. The construction of conformal blocks in terms of chiral vertex operators [8][9] identifies the *fusion-coefficients* $F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ as analogues of the Racah-Wigner or 6j-coefficients for fusion-products of representations of the Virasoro-algebra.

We will now restrict attention to $\alpha_i \in \mathbb{S}$ until the end of this section. This will not be a loss of generality since it will turn out that the general case can be obtained by analytic continuation. The requirement can then be rewritten as the system of equations

$$
\int_{\mathbb{S}} d\alpha_{21}\; C(\alpha_4,\alpha_3,\alpha_{21})C(\bar{\alpha}_{21},\alpha_2,\alpha_1)F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\left(F_{\alpha_{21}\beta_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^* = \tag{8}
$$
$$
= \delta(\alpha_{32}-\beta_{32})C(\alpha_4,\alpha_{32},\alpha_1)C(\bar{\alpha}_{32},\alpha_3,\alpha_2).
$$

[4]The superscript "s" and "t" refer to s- and t-channel respectively

This may be brought into a more suggestive form by absorbing (part of) the factors $C(\alpha_3,\alpha_2,\alpha_1)$ by a change of normalisation of the conformal blocks (which is in fact a change of normalisation of the chiral vertex operators): Let

$$\begin{aligned}\mathcal{F}^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z}) &= N(\alpha_4,\alpha_3,\alpha_{21})N(\alpha_{21},\alpha_2,\alpha_1)\mathcal{G}^s_{\alpha}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z})\\ \mathcal{F}^t_{\alpha}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z}) &= N(\alpha_4,\alpha_{32},\alpha_1)N(\alpha_{32},\alpha_3,\alpha_2)\mathcal{G}^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{z}),\end{aligned} \tag{9}$$

where the following choice of $N(\alpha_3,\alpha_2,\alpha_1)$ will turn out to be convenient:

$$\begin{aligned}&N(\alpha_3,\alpha_2,\alpha_1) =\\ &=\frac{\Gamma_b(2\alpha_1)\Gamma_b(2\alpha_2)\Gamma_b(2Q-2\alpha_3)}{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\Gamma_b(\alpha_1+\alpha_2-\alpha_3)\Gamma_b(\alpha_1+\alpha_3-\alpha_2)\Gamma_b(\alpha_2+\alpha_3-\alpha_1)},\end{aligned} \tag{10}$$

where $\Gamma_b(x)$ is essentially the double Gamma function of Barnes [22], see the Appendix. The blocks $\mathcal{G}^s_\alpha$ and $\mathcal{G}^t_\alpha$ will then be related by an equation of the form (7) with $F_{\alpha_{21}\alpha_{32}}$ replaced by

$$G_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{N(\alpha_4,\alpha_{32},\alpha_1)N(\alpha_{32},\alpha_3,\alpha_2)}{N(\alpha_4,\alpha_3,\alpha_{21})N(\alpha_{21},\alpha_2,\alpha_1)}\,F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]. \tag{11}$$

The locality condition now takes the form

$$\int_{\mathbb{S}} d\alpha_{21}\;\; M_b(\alpha_{21})\;\; G_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\left(G_{\alpha_{21}\beta_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]\right)^* \;=\; M_b(\alpha_{32})\;\;\delta(\alpha_{32}-\beta_{32}), \tag{12}$$

where $M_b(\alpha) = -4\sin(\pi b(2\alpha-Q))\sin(\pi b^{-1}(2\alpha-Q))$. Equation (12) expresses *unitarity* of the change of basis (7) when the space of conformal blocks spanned by $\{\mathcal{F}^s_{\alpha_{21}};\alpha_{21}\in\mathbb{S}\}$ is equipped with the Hilbert-space structure $\langle\mathcal{F}^s_{\alpha_{21}},\mathcal{F}^s_{\beta_{21}}\rangle = (M_b(\alpha_{21}))^{-1}\delta(\alpha_{21}-\beta_{21})$.

It is the aim of the present work to verify (12) by (a) showing that the coefficients $G_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ are given by Racah-Wigner coefficients for a category of representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ and furthermore (b) deriving the relevant orthogonality relations from completeness of the Clebsch-Gordan decomposition for tensor products of these representations.

3. DIFFERENCE EQUATIONS FOR THE FUSION COEFFICIENTS

It will be assumed that the conformal blocks for any N-point function can be constructed in terms of chiral vertex operators. One therefore finds as in [8][9] that the fusion coefficients have to satisfy a system of consistency conditions called hexagon and pentagon equations. Here we will be particularly interested in the pentagon equation which takes the form

$$\int_{\mathbb{S}} d\delta_1\;F_{\beta_1\delta_1}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \gamma_2 & \alpha_1\end{smallmatrix}\right]F_{\beta_2\gamma_2}\left[\begin{smallmatrix}\alpha_4 & \delta_1\\ \alpha_5 & \alpha_1\end{smallmatrix}\right]F_{\delta_1\gamma_1}\left[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \gamma_2 & \alpha_2\end{smallmatrix}\right] = F_{\beta_2\gamma_1}\left[\begin{smallmatrix}\alpha_4 & \alpha_3\\ \alpha_5 & \beta_1\end{smallmatrix}\right]F_{\beta_1\gamma_2}\left[\begin{smallmatrix}\alpha_5 & \gamma_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\right] \tag{13}$$

The first crucial observation to be made at this point is that the fusion coefficients should have certain analyticity properties in the dependence on its six complex parameters. Note that the coefficients $\mathcal{F}^s_{\alpha_{21},n}$ that appear in the power series representation of conformal blocks (3) depend *polynomially* on the variables $\alpha_4,\ldots,\alpha_1$ and rationally on α, with poles only at the values $2\alpha = \alpha_{m,n} = -mb-nb^{-1}$ and $Q-\alpha = \alpha_{m,n}$. If the series (3) converges[5] it follows that the dependence of $\mathcal{F}^s_{\alpha_{21}}$ on the variables $\alpha_4,\ldots,\alpha_1$ must be entire analytic, whereas the α-dependence will be meromorphic with poles only at the locations given previously. The compatibility of the

[5] Liouville theory would be dead alltogether otherwise

existence of fusion transformations (7), analyticity properties of conformal blocks and Remark 1 suggests (see [18] for more discussion) the following conjecture:

Conjecture 2. The fusion coefficients $F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ are holomorphic in

$$
\begin{aligned}
&0 < \mathrm{Re}(\alpha_2+\alpha_3+\alpha_{23}-Q) < Q \qquad && 0 < \mathrm{Re}(\alpha_4+\alpha_1+\alpha_{23}-Q) < Q\\
&0 < \mathrm{Re}(\alpha_2+\alpha_3-\alpha_{23}) < Q && 0 < \mathrm{Re}(\alpha_4+\alpha_1-\alpha_{23}) < Q\\
&0 < \mathrm{Re}(\alpha_{23}+\alpha_3-\alpha_2) < Q && 0 < \mathrm{Re}(\alpha_{23}+\alpha_4-\alpha_1) < Q\\
&0 < \mathrm{Re}(\alpha_{23}+\alpha_2-\alpha_3) < Q && 0 < \mathrm{Re}(\alpha_{23}+\alpha_1-\alpha_4) < Q
\end{aligned}
\tag{14}
$$

Second, one may observe that the fusion transformations (7) simplify if one of $\alpha_4,\dots,\alpha_1$ is taken to be $\alpha=\alpha_{m,n}=-mb-nb^{-1}$, corresponding to a degenerate representation of the Virasoro algebra. Conformal blocks $\mathcal{F}^s_{\alpha_{21}}$, $\mathcal{F}^t_{\alpha_{32}}$ then only exist for a finite number of values of α_{21}, α_{32}, so that the fusion coefficients form a finite dimensional matrix [20]. It will suffice to consider cases where one of $\alpha_4,\dots,\alpha_1$, say α_2, equals $-b$ or $-b^{-1}$. In that case s-channel conformal blocks $\mathcal{F}^s_{\alpha_{21}}$ exist only for $\alpha=\alpha_1-sb$, $s=-,0,+$, and t-channel conformal blocks for $\alpha=\alpha_3-sb$, $s=-,0,+$. Moreover, it is easy to show that the fusion coefficients that appear in such cases can all be uniquely expressed in terms of the following "elementary" ones: Let

$$
F_{s,s'}(\alpha_4,\alpha_3,\alpha_1)\equiv F_{\alpha_1-s\frac{b}{2},\alpha_3-s'\frac{b}{2}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]_{\alpha_2=-\frac{b}{2}}, \qquad \text{where} \quad s,s'=+,-
\tag{15}
$$

The matrix $F_{s,s'}$ can be calculated explicitly [19][20]. The corresponding matrix $G_{s,s'}$ (cf. (11)) is then given by

$$
\begin{aligned}
G_{++}&=\frac{[\alpha_4+\alpha_3-\alpha_1-\frac{b}{2}]}{[2\alpha_3-b]} \qquad & G_{+-}&=\frac{[\alpha_4+\alpha_3+\alpha_1-\frac{3b}{2}]}{[2\alpha_3-b]}\\
G_{-+}&=\frac{[\alpha_3+\alpha_1-\alpha_4-\frac{b}{2}]}{[2\alpha_3-b]} & G_{--}&=-\frac{[\alpha_4+\alpha_1-\alpha_3-\frac{b}{2}]}{[2\alpha_3-b]}
\end{aligned}
\qquad [x]\equiv\frac{\sin(\pi b x)}{\sin(\pi b^2)}.
\tag{16}
$$

If one then considers the pentagon equation (13) in the special cases where one of $\alpha_5,\dots,\alpha_1$ equals $-b$, one finds a set of *linear finite difference* equations for the general fusion coefficients $F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$. Part of these equations involve shifts of one argument only, for example

$$
\sum_{s=-,0,+} C_s\left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\alpha_{21}\\ \alpha_{32}\end{smallmatrix}\right)F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1-sb\end{smallmatrix}\right]=0.
\tag{17}
$$

One has one such equations of each of the variables $\alpha_4,\dots\alpha_1$. Other equations are of the form

$$
\sum_{s=-,0,+} D_{r,s}\left(\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\,\middle|\,\begin{smallmatrix}\alpha_{21}\\ \alpha_{32}\end{smallmatrix}\right)F_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1-sb\end{smallmatrix}\right]=F_{\alpha_{21}+rb,\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1-sb\end{smallmatrix}\right] \quad \text{where } r=+,-,
\tag{18}
$$

and a similar equation with shifts of α_{32} on the right hand side. Furthermore, each of these equations has a "dual" partner obtained by $b\to b^{-1}$. Finally, one has equations that reflect the fact that all the fusion coefficients $F_{\beta_1,\beta_2}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ are functions of the conformal dimensions only, so must be unchanged under $\alpha_i\to Q-\alpha_i$, $i=1,2,3,4$ and $\beta_j\to Q-\beta_j$, $j=1,2$.

In the case of real irrational b it is possible to show (details will appear in [18]) uniqueness of a solution to this system of functional equations, taking into account the analytic properties of the fusion coefficients. In fact, the equations (17) are second order homogeneous finite difference equations. It can be shown that the second order equations of the form (17) together with their $b\to b^{-1}$ duals can have at most two linearly independent solutions with the required analytic properties. Taking into account the symmetry $\alpha_i\to Q-\alpha_i$, $i=1,\dots,4$ will determine the dependence w.r.t. α_i,

$i = 1, \ldots, 4$ up to a factor that depends on α_{21} and α_{32}. The remaining freedom is then fixed by considering equations (18) and its counterpart with shifts of α_{32}.

Remark 3. Loosely speaking the message is the following: If there exist fusion transformations of conformal blocks that are compatible with the expectations from other approaches (encoded in Conjectures 1 and 2; cf. Remark 1) then they are unique for real, irrational b.

Remark 4. One might also be interested in the possibility of having fusion transformations of the form (7) but with coefficients $F_{\alpha_{21},\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ only required to be defined for real $\alpha_4, \ldots, \alpha_1$ *and* α_{21}, α_{32}. But if one then only requires e.g. continuity in some interval such as (14) one still has the above result on uniqueness, which together with the results to be discussed below put one back precisely into the situation considered here.

4. A TENSOR CATEGORY OF QUANTUM GROUP REPRESENTATIONS

A set of infinite dimensional representations $\mathcal{P}_\alpha$ of the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ may be realized on the Hilbert space $L^2(\mathbb{R})$ in terms of the Weyl-algebra generated by $U = e^{2\pi b x}$ and $V = e^{-\frac{b}{2}p}$, where $[x,p] = i$:

$$
(19)\qquad
\begin{aligned}
E &= U^{+1}\frac{e^{\pi i b(Q-\alpha)}V - e^{-\pi i b(Q-\alpha)}V^{-1}}{e^{\pi i b^2} - e^{-\pi i b^2}}\\
F &= U^{-1}\frac{e^{-\pi i b(Q-\alpha)}V - e^{\pi i b(Q-\alpha)}V^{-1}}{e^{\pi i b^2} - e^{-\pi i b^2}}
\end{aligned}
\qquad K = V
$$

These generators satisfy the relations

$$
(20)\qquad KE = qEK \qquad KF = q^{-1}FK \qquad [E,F] = -\frac{K^2 - K^{-2}}{q - q^{-1}}
$$

The operators E, F, K are unbounded. They will be defined on domains consisting of functions which possess an analytic continuation into the strip $\{x \in \mathbb{C}; |\mathrm{Im}(x)| < \frac{b}{2}\}$ and which have suitable decay properties at infinity. It may be shown that the representations $\mathcal{P}_\alpha$, $\alpha \in \mathbb{S}$ are unitarily equivalent to a subset of the integrable ("well-behaved") representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ classified in [23]. It follows in particular that E, F and K become self-adjoint on suitable domains.

The q-Casimir acts as a scalar in this representation:

$$
(21)\qquad C = FE - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q-q^{-1})^2} \equiv [\alpha - \tfrac{Q}{2}]^2, \qquad [x] \equiv \frac{\sin(\pi b x)}{\sin(\pi b^2)}.
$$

Remark 5. The representations considered here form a subset of the representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ that appear in the classification of [23]. This subset is distinguished by the fact that it is simultaneously a representation of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, $\tilde{q} = \exp(\pi i b^{-2})$ with generators $\tilde{E}$, $\tilde{F}$, $\tilde{K}$ being realized by replacing $b \to b^{-1}$ in the expressions for E, F, K given above. Restriction to these representations is crucial for obtaining a quantum group structure which is self-dual under $b \to b^{-1}$, as Liouville theory is. The price to pay is that the representations $\mathcal{P}_\alpha$ do not have classical ($b \to 0$) counterparts.

Tensor products $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ of representations can be defined by means of the co-product:

$$
(22)\qquad \Delta(K) = K \otimes K \qquad \Delta(E) = E \otimes K^{-1} + K \otimes E \qquad \Delta(F) = F \otimes K^{-1} + K \otimes F
$$

Theorem 1. *The $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$-representation $\mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ defined on $L^2(\mathbb{R}^2)$ by means of Δ decomposes as follows*

$$
(23)\qquad \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \simeq \int_{\mathbb{S}}^{\oplus} d\alpha\; \mathcal{P}_\alpha,
$$

It is remarkable and nontrivial that the subset of "self-dual" integrable representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ is actually closed under tensor products. The proof, based on results and techniques of [16], will appear in [17].

The Clebsch-Gordan maps $C(\alpha_3|\alpha_2,\alpha_1) : \mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}\to\mathcal{P}_{\alpha_3}$ may be explicitly represented as an integral transform

$$C(\alpha_3|\alpha_2,\alpha_1) : f(x_2,x_1) \longrightarrow F[f](\alpha_3|x_3) \equiv \int_{\mathbb{R}} dx_2 dx_1 \left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right] f(x_2,x_1). \tag{24}$$

The distributional kernel $[\dots]$ (the "Clebsch-Gordan coefficients") is given by the expression

$$\begin{aligned}\left[\begin{smallmatrix}Q-\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right] =& S_b(\alpha_1+\alpha_2-\alpha_3)e^{\pi i\alpha_1\alpha_2}\; e^{2\pi(x_3(\alpha_2-\alpha_1)-\alpha_2x_2+\alpha_1x_1)}\;\cdot\\ &\cdot D_b(x_{32},\alpha_{32})D_b(x_{31},\alpha_{31})D_b(x_{21},\alpha_{21})\end{aligned} \tag{25}$$

where the distribution $D_b(x,\alpha)$ is defined in terms of the Double Sine function $S_b(x)$ (cf. Appendix) as

$$D_b(x,\alpha) = e^{-\frac{\pi i}{2}\alpha(\alpha-Q)}e^{\pi\alpha x}\lim_{\epsilon\to 0+}\frac{S_b(ix+\epsilon)}{S_b(ix+\alpha)} \tag{26}$$

and the coefficients x_{ji}, α_{ji}, $j>i\in\{1,2,3\}$ are given by

$$\begin{aligned} x_{32} &= x_3-x_2+\tfrac{i}{2}(\alpha_3+\alpha_2-Q) \qquad & \alpha_{32} &= \alpha_2+\alpha_3-\alpha_1\\ x_{31} &= x_3-x_1+\tfrac{i}{2}(\alpha_3+\alpha_1-Q) & \alpha_{31} &= \alpha_3+\alpha_1-\alpha_2\\ x_{21} &= x_2-x_1+\tfrac{i}{2}(\alpha_2+\alpha_1-2\alpha_3) & \alpha_{21} &= \alpha_2+\alpha_3-\alpha_1.\end{aligned} \tag{27}$$

The orthogonality relations for the Clebsch-Gordan coefficients (25) can be determined by explicit calculation:

$$\int\limits_{\mathbb{R}} dx_1dx_2\, \left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right]^*\left[\begin{smallmatrix}\beta_3 & \alpha_2 & \alpha_1\\ y_3 & x_2 & x_1\end{smallmatrix}\right] = |S_b(2\alpha_3)|^{-2}\delta(\alpha_3-\beta_3)\delta(x_3-y_3). \tag{28}$$

Together with Theorem 1 one obtains the corresponding completeness relations

$$\int\limits_{\mathbb{S}} d\alpha_3\, |S_b(2\alpha_3)|^2\int\limits_{\mathbb{R}} dx_3\, \left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & x_2 & x_1\end{smallmatrix}\right]^*\left[\begin{smallmatrix}\alpha_3 & \alpha_2 & \alpha_1\\ x_3 & y_2 & y_1\end{smallmatrix}\right] = \delta(x_2-y_2)\delta(x_1-y_1). \tag{29}$$

The braiding-operation $B:\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}\to\mathcal{P}_{\alpha_1}\otimes\mathcal{P}_{\alpha_2}$ may then be introduced as [6]

$$B_{21} = \int\limits_{\mathbb{S}} d\alpha_3\, |S_b(2\alpha_3)|^2\; C^\dagger(\alpha_1,\alpha_2|\alpha_3)\,\Omega\left(\begin{smallmatrix} & \alpha_3 & \\ \alpha_2 & & \alpha_1\end{smallmatrix}\right)\, C(\alpha_3|\alpha_2,\alpha_1), \tag{30}$$

where $C^\dagger(\alpha_1,\alpha_2|\alpha_3) : \mathcal{P}_{\alpha_3}\to\mathcal{S}_{21}'$ is the adjoint of $C(\alpha_3|\alpha_2,\alpha_1)$ for any Gelfand-triple $\mathcal{S}_{21}\subset\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}\subset\mathcal{S}_{21}'$, and

$$\Omega\left(\begin{smallmatrix} & \alpha_3 & \\ \alpha_2 & & \alpha_1\end{smallmatrix}\right) = e^{\pi i(\alpha_3(Q-\alpha_3)-\alpha_2(Q-\alpha_2)-\alpha_1(Q-\alpha_1))}. \tag{31}$$

Triple tensor products $\mathcal{P}_{\alpha_3}\otimes\mathcal{P}_{\alpha_2}\otimes\mathcal{P}_{\alpha_1}$ carry a representation of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ given by $(\mathrm{id}\otimes\Delta)\circ\Delta = (\Delta\otimes\mathrm{id})\circ\Delta$. The projections affecting the decomposition of this representation into irreducibles can be constructed by iterating Clebsch-Gordan maps. One thereby obtains two canonical bases in

[6] L. Faddeev has explained to the authors a nice alternative method to introduce a R-operator.

the sense of generalized eigenfunctions for $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ given by the sets of distributions ($\mathfrak{x} = (x_4, \dots, x_1)$)

$$
\begin{aligned}
\Phi^s_{\alpha_{21}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{x}) &= \int_{\mathbb{R}} dx_{21}\; \left[\begin{smallmatrix} \alpha_4 & \alpha_3 & \alpha_{21} \\ x_4 & x_3 & x_{21} \end{smallmatrix}\right]\left[\begin{smallmatrix} \alpha_{21} & \alpha_2 & \alpha_1 \\ x_{21} & x_2 & x_1 \end{smallmatrix}\right] \quad \alpha_4, \alpha_{21} \in \mathbb{S}, x_4 \in \mathbb{R} \\
\Phi^t_{\alpha_{32}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{x}) &= \int_{\mathbb{R}} dx_{32}\; \left[\begin{smallmatrix} \alpha_4 & \alpha_{32} & \alpha_1 \\ x_4 & x_{32} & x_1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \alpha_{32} & \alpha_3 & \alpha_2 \\ x_{32} & x_3 & x_2 \end{smallmatrix}\right]. \quad \alpha_4, \alpha_{32} \in \mathbb{S}, x_4 \in \mathbb{R}
\end{aligned} \tag{32}
$$

In particular one may observe that $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$ can be decomposed into eigenspaces of the pair of commuting self-adjoint operators $\pi_{321}(C)$ and $\pi_{321}(K)$ as follows:

$$
\mathcal{P}_{a_3} \otimes \mathcal{P}_{a_2} \otimes \mathcal{P}_{a_1} \simeq \int_{\mathbb{S}}^{\oplus} d\alpha \int_{\mathbb{R}}^{\oplus} dk\; \mathcal{H}_{\alpha,k}. \tag{33}
$$

It then follows from completeness of the bases $\mathfrak{B}^s_{321}$ and $\mathfrak{B}^t_{321}$ and orthogonality of the eigenspaces $\mathcal{H}_{\alpha,k}$ that the bases Φ^s and Φ^t must be related by a transformation of the form

$$
\Phi^s_{\alpha_{21}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{x}) \;=\; \int_{\mathbb{S}} d\alpha_{32}\; \left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\middle|\begin{smallmatrix} \alpha_{21} \\ \alpha_{32} \end{smallmatrix}\right\}_b \; \Phi^t_{\alpha_{32}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right](\mathfrak{x}) \tag{34}
$$

thereby defining the b-Racah-Wigner symbols $\left\{\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}\middle|\begin{smallmatrix} \cdot \\ \cdot \end{smallmatrix}\right\}_b$.

The data $\Omega\left(\begin{smallmatrix} & \alpha_3 & \\ \alpha_2 & & \alpha_1 \end{smallmatrix}\right)$ and $\left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_3 \end{smallmatrix}\middle|\begin{smallmatrix} \alpha_{21} \\ \alpha_{32} \end{smallmatrix}\right\}_b$ will now satisfy all the Moore-Seiberg consistency conditions.

Moreover, by again using completeness of the bases $\mathfrak{B}^s_{321}$ and $\mathfrak{B}^t_{321}$ one finds the following orthogonality relations for the b-Racah-Wigner symbols

$$
\int_{\mathbb{S}} d\alpha_{21}\; |S_b(2\alpha_{21})|^2 \left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\middle|\begin{smallmatrix} \alpha_{21} \\ \alpha_{32} \end{smallmatrix}\right\}_b \left(\left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\middle|\begin{smallmatrix} \alpha_{21} \\ \beta_{32} \end{smallmatrix}\right\}_b\right)^* \;=\; |S_b(2\alpha_{32})|^2\; \delta(\alpha_{32} - \beta_{32}). \tag{35}
$$

In fact, $|S_b(2\alpha)|^2 \;=\; M_b(\alpha)$, so that (35) is indeed the orthogonality relation required to prove locality or crossing symmetry in Liouville theory once equality of $G_{\alpha_{21},\alpha_{32}}\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{smallmatrix}\right]$ and $\left\{\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_3 \end{smallmatrix}\middle|\begin{smallmatrix} \alpha_{21} \\ \alpha_{32} \end{smallmatrix}\right\}_b$ will be established.

5. Space of functions on the quantum group

The following section may be skipped by readers interested mainly in Liouville theory. It is important, however, for the deeper mathematical understanding why there is such a remarkable non-classical category of representations. An explanation can be given by constructing the associated dual object (a suitable space of functions on the corresponding quantum group) and studying its harmonic analysis. Details will be given in [16].

Define operators A, B, C, D on $L^2(\mathbb{R} \times \mathbb{R})$ by the expressions

$$
\begin{aligned}
A &= U_1^{-1} & B &= V_1 U_2 & U_j &= e^{2\pi b x_j} & & \\
C &= V_1 & D &= U_1(1 + qU_2V_1^2). & V_j &= e^{-\frac{b}{2}p_j} & [x_i, p_j] &= i\delta_{i,j} \qquad j = 1,2
\end{aligned} \tag{36}
$$

This is known to generate an integrable operator representation of the algebra $\mathrm{Pol}(SL_q(2,\mathbb{R}))$ [23]. In particular, the operators A, B, C, D all have self-adjoint extensions and one has the nontrivial relations

$$
\begin{aligned}
AB &= qBA & DB &= q^{-1}BD & AD - qBC &= 1 \\
AC &= qCA & DC &= q^{-1}CD & AD - DA &= (q - q^{-1})BC.
\end{aligned} \tag{37}
$$

Again it is worth noting that restriction to the integrable representations of $\mathrm{Pol}(SL_q(2,\mathbb{R}))$ from [23] in which A, B, C, D have positive spectrum already represents a point of departure from classical $SL(2,\mathbb{R})$ [7], but has the desired feature of being simultaneously a representation of the $(b \to b^{-1})$-dual algebra $\mathrm{Pol}(SL_{\tilde{q}}(2,\mathbb{R}))$. The algebra generated by *positive selfadjoint* Hilbert space operators A, B, C, D with relations (37) will be denoted $\mathrm{Pol}(SL_q^+(2,\mathbb{R}))$.

Let $\mathcal{A}$ be the norm closure of the set $\mathcal{C}_c^\infty(SL_q^+(2,\mathbb{R}))$ of all operators on $L^2(\mathbb{R}\times\mathbb{R})$ of the form

$$\mathcal{O}[f] = \int_{\mathbb{R}} drds\ A^{\frac{ir}{b}}\, C^{\frac{is}{b}} f(r,s|x)\, C^{\frac{is}{b}}\, A^{\frac{ir}{b}}, \qquad x \equiv \tfrac{1}{2\pi b}\log(BC) \tag{38}$$

where the so-called *symbol* $f(r,s|x)$ is smooth and of compact support in its dependence of the variable x, and of Paley-Wiener class (entire analytic, rapid decay) w.r.t. the variables r and s. $\mathcal{A}$ is then a nonunital C^*-algebra generated by elements A, B, C, D affiliated with $\mathcal{A}$ in the sense of Woronowicz [25, 26].

On the Hopf-algebra $\mathrm{Pol}(SL_q(2,\mathbb{R}))$ one has natural analogues of left- and right regular representation defined by means of the duality between $\mathrm{Pol}(SL_q(2,\mathbb{R}))$ and $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$, see e.g. [27]. There is a corresponding action on $\mathcal{C}_c^\infty(SL_q^+(2,\mathbb{R}))$ that may be expressed conveniently in terms of finite difference operators acting on the symbols $f(r,s|x)$:

$$\begin{aligned} E_l &= T_r^+ T_s^+ e^{-2\pi b x}[\delta_x] \\ F_l &= T_r^- T_s^-([\delta_x + 2is] + e^{2\pi b x}[\delta_x + 2i(s+r)]) \qquad & K_l &= e^{-\pi(r+s)} \\ E_r &= T_r^- T_s^+([\delta_x + 2ir] + e^{-2\pi b x}[\delta_x]) & K_r &= e^{-\pi(r-s)}, \\ F_r &= T_r^+ T_s^-[\delta_x + 2is] \end{aligned} \tag{39}$$

where the following notation has been used: $[x] \equiv \frac{\sin(\pi b x)}{\sin(\pi b^2)}$ and

$$T_r^\pm f(r,s|x) = f(r \pm \tfrac{ib}{2}, s|x) \qquad T_s^\pm f(r,s|x) = f(r, s \pm \tfrac{ib}{2}|x) \qquad \delta_x = \tfrac{1}{2\pi}\partial_x. \tag{40}$$

A L^2-space may be introduced as completion of $\mathcal{C}_c^\infty(SL_q^+(2,\mathbb{R}))$ with respect to the inner product $\langle .,. \rangle$ defined as

$$\langle \mathcal{O}[f_1], \mathcal{O}[f_2] \rangle = \int_{\mathbb{R}} drds \int_{\mathbb{R}} dx\ e^{2\pi Q x} \big(f_1(r,s|x)\big)^* f_2(r,s|x). \tag{41}$$

This inner product is such that the operators E_l, F_l, K_l and E_r, F_r, K_r are symmetric.

Theorem 2. *One has the following decomposition of $L^2(SL_q^+(2,\mathbb{R}))$ into irreducible representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))_l \otimes \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))_r$:*

$$L^2(SL_q^+(2,\mathbb{R})) \simeq \int_{\mathbb{S}}^{\oplus} d\alpha\ |S_b(2\alpha)|^2\ \mathcal{P}_\alpha \otimes \mathcal{P}_\alpha \tag{42}$$

One of the authors (J.T.) has partial results that strongly support the conjecture that the co-product exists on the Hilbert-space level in the sense of [25].

Remark 6. One obtains representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))_l \otimes \mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))_r$ by symmetric operators E_l, F_l, K_l and E_r, F_r, K_r also if one chooses the measure in (41) to be $\exp(2\pi(b+\frac{k}{b})x)$. The choice $k=0$ in particular reproduces the Haar-measure on classical $SL(2,\mathbb{R})$. For the present choice $k=1$ one looses the correspondence to any classical object but gains the self-duality $b \to b^{-1}$.

[7] The spectrum does not cover the group manifold.

6. Calculation of Racah coefficients

The direct calculation of the Racah coefficients from the Clebsch-Gordan maps looks difficult. A small trick helps. If one fixes the values of three of the four variables $x_4, \dots, x_1$ in (34) one obtains an integral transformation for a function of a single variable. In fact, the analytic properties of $\Phi^s_{\alpha_{21}}$ and $\Phi^t_{\alpha_{32}}$ even allow to choose complex values. It will be convenient to consider

$$\Psi^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = \lim_{x_4\to\infty} e^{2\pi\alpha_4 x_4} \lim_{x_2\to-\infty} \prod_{j=1}^{3} e^{-2\pi\alpha_j x_j}\, \Phi^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](\mathfrak{x})\Big|^{x_1=x}_{x_3=\frac{i}{2}(Q+\alpha_2-\alpha_4)}, \tag{43}$$

and the same for $\Psi^t_{\alpha_{32}}$. The integral that defines $\Phi^s_{\alpha_{21}}$ and $\Phi^t_{\alpha_{32}}$ according to (32),(43) can be done explicitly by using (59). One finds expressions of the form

$$\begin{aligned}
&\Psi^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = N^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]\Theta^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\\
&\qquad\Theta^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = e^{+2\pi x(\alpha_{21}-\alpha_2-\alpha_1)} F_b(\alpha_{21}+\alpha_1-\alpha_2, \alpha_{21}+\alpha_3-\alpha_4; 2\alpha_{21}; -ix)\\
&\Psi^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = N^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]\Theta^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\\
&\qquad\Theta^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = e^{-2\pi x(\alpha_{32}+\alpha_1-\alpha_4)} F_b(\alpha_{32}+\alpha_3-\alpha_2, \alpha_{32}+\alpha_1-\alpha_4; 2\alpha_{32}; +ix),
\end{aligned} \tag{44}$$

where F_b is the b-hypergeometric function defined in the Appendix, and $N^s_{\alpha_{21}}$, $N^t_{\alpha_{32}}$ are certain normalization factors.

The linear transformation following from (34) can now be calculated as follows: One observes that $\Psi^s_{\alpha_{21}}$ (resp. $\Psi^t_{\alpha_{32}}$) are eigenfunctions of the finite difference operators $\mathcal{C}_{21}$ and $\mathcal{C}_{32}$ defined respectively by

$$\begin{aligned}
\mathcal{C}_{21} &= \left[\delta_x + \alpha_1 + \alpha_2 - \tfrac{Q}{2}\right]^2 - e^{+2\pi b x}\left[\delta_x + \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4\right]\left[\delta_x + 2\alpha_1\right]\\
\mathcal{C}_{32} &= \left[\delta_x + \alpha_1 - \alpha_4 + \tfrac{Q}{2}\right]^2 - e^{-2\pi b x}\left[\delta_x + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4\right]\left[\delta_x\right],
\end{aligned} \tag{45}$$

where $\delta_x = (2\pi)^{-1}\partial_x$. These operators can be made self-adjoint in $L^2(\mathbb{R}, dx e^{2\pi Q x})$, and it can be shown that

Theorem 3. $\{\Theta^s_{\alpha_{21}}; \alpha_{21} \in \mathbb{S}\}$ *and* $\{\Theta^t_{\alpha_{32}}; \alpha_{32} \in \mathbb{S}\}$ *form complete sets of eigenfunctions of the operators* $\mathcal{C}_{21}$ *and* $\mathcal{C}_{32}$ *respectively, normalized by*

$$\int_{\mathbb{R}} dx\, e^{2\pi Q x} \left(\Theta^s_{\alpha'_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\right)^* \Theta^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x) = \delta(\alpha_{21} - \alpha'_{21}). \tag{46}$$

It follows that the Racah-Wigner coefficients can be evaluated in terms of the overlap between these two bases:

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \bar{\alpha}_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_{21}\\ \alpha_{32}\end{smallmatrix}\right\}_b = \frac{N^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]}{N^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]} \int_{\mathbb{R}} dx\, e^{2\pi Q x} \left(\Theta^t_{\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x)\right)^* \Theta^s_{\alpha_{21}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right](x). \tag{47}$$

The integral can be done by using the representation (57) for the b-hypergeometric function. The result is

$$
\begin{aligned}
&\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \bar{\alpha}_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_{21} \\ \alpha_{32} \end{smallmatrix} \right\}_b = \\
&\frac{S_b(\alpha_2+\alpha_{21}-\alpha_1)S_b(\alpha_2+\alpha_1-\alpha_{21})S_b(\alpha_{21}+\alpha_3+\alpha_4-Q)S_b(\alpha_{32}+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_{32}-\alpha_3)S_b(\alpha_3+\alpha_2-\alpha_{32})S_b(\alpha_{21}+\alpha_3-\alpha_4)S_b(\alpha_{32}+\alpha_1+\alpha_4-Q)} \cdot \\
&\cdot |S_b(2\alpha_{32})|^2 \int_{-i\infty}^{i\infty} ds\ \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(V_4+s)},
\end{aligned} \tag{48}
$$

where the coefficients U_i and V_i, $i = 1, \dots, 4$ are given by

$$
\begin{aligned}
U_1 &= \alpha_{21}+\alpha_1-\alpha_2 & V_1 &= 2Q+\alpha_{21}-\alpha_{32}-\alpha_2-\alpha_4 \\
U_2 &= Q+\alpha_{21}-\alpha_2-\alpha_1 & V_2 &= Q+\alpha_{21}+\alpha_{32}-\alpha_4-\alpha_2 \\
U_3 &= \alpha_{21}+\alpha_3-\alpha_4 & V_3 &= 2\alpha_{21} \\
U_4 &= Q+\alpha_{21}-\alpha_3-\alpha_4 & V_4 &= Q.
\end{aligned} \tag{49}
$$

The integral representing the Racah-Wigner coefficients is of the type of a Barnes integral for a ${}_4F_3$ b-hypergeometric function.

7. FUNCTIONAL EQUATIONS FOR RACAH-WIGNER COEFFICIENTS

There is a family of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$ labelled by two positive integers n, m. The generators E, F, K and $\tilde{E}, \tilde{F}, \tilde{K}$ are realized on vector spaces $\mathcal{P}_{n,m}$ of polynomials in the variables U and $\tilde{U} = e^{2\pi b^{-1}x}$ by means of the restriction of the expressions (19) to $Q - \alpha = -\frac{n}{2}b - \frac{m}{2}b^{-1}$, $n, m = 0, 1, 2, \dots$. Of particular interest will be the pair of two-dimensional representations $\mathcal{P}_{1,0}$ and $\mathcal{P}_{0,1}$, from which all representations $\mathcal{P}_{n,m}$ can be generated by repeated tensor products. The decomposition of tensor products of representations $\mathcal{P}_{1,0}$ or $\mathcal{P}_{0,1}$ with a generic representation $\mathcal{P}_\alpha$ into irreducible representations can be determined purely algebraically:

$$
\mathcal{P}_{1,0} \otimes \mathcal{P}_\alpha \simeq \bigoplus_{s=\pm} \mathcal{P}_{\alpha - s\frac{b}{2}} \qquad \mathcal{P}_{0,1} \otimes \mathcal{P}_\alpha \simeq \bigoplus_{s=\pm} \mathcal{P}_{\alpha - s\frac{1}{2b}} \tag{50}
$$

The distributions $\Psi^s_{\alpha_{21}}$ (resp. $\Psi^t_{\alpha_{32}}$) develop double poles if one sets e.g. $\alpha_2 = -\frac{b}{2}$, $\alpha_{21} = \alpha_1 - \sigma\frac{b}{2}$, $\sigma = +, -$ (resp. $\alpha_{32} = \alpha_3 - \sigma\frac{b}{2}$ and $b \to b^{-1}$). The relevant coefficients describing tensoring with the finite dimensional representations $\mathcal{P}_{1,0}$ (resp. $\mathcal{P}_{0,1}$) are found as residues of these double poles. These residues will be denoted as Ψ^s_σ (resp. Ψ^t_σ) and are given by the expressions (let $\mathrm{s}(x) \equiv 2\sin(\pi b x)$, $z = e^{2\pi b x}$)

$$
\begin{aligned}
\Psi^s_+\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \bar{\alpha}_4 & \alpha_1 \end{smallmatrix}\right](x) &= R(x)\ \left([2\alpha_1 - b] + [\alpha_1+\alpha_4-\alpha_3-\tfrac{b}{2}]z\right) \\
\Psi^s_-\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \bar{\alpha}_4 & \alpha_1 \end{smallmatrix}\right](x) &= R(x)\ [\alpha_4+\alpha_3+\alpha_1-\tfrac{3b}{2}-b^{-1}]z \\
\Psi^t_+\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \bar{\alpha}_4 & \alpha_1 \end{smallmatrix}\right](x) &= R(x)\ \left([2\alpha_3 - b]z + [\alpha_3+\alpha_4-\alpha_1-\tfrac{b}{2}]\right) \\
\Psi^t_-\left[\begin{smallmatrix} \alpha_3 & \alpha_2 \\ \bar{\alpha}_4 & \alpha_1 \end{smallmatrix}\right](x) &= R(x)\ [\alpha_4+\alpha_3+\alpha_1-\tfrac{3b}{2}-b^{-1}],
\end{aligned} \tag{51}
$$

where $R(x)$ abbreviates the common factor that appears. It now follows easily that the matrix $G_{s,t}$ introduced in Section 3 coincides with the Racah coefficients that relate Ψ^s_σ and Ψ^t_σ:

$$
G_{s,t}(\alpha_4, \alpha_3, \alpha_1) \equiv \left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix} \middle| \begin{smallmatrix} \alpha_1 - s\alpha_2 \\ \alpha_3 - t\alpha_2 \end{smallmatrix} \right\}_b \Bigg|_{\alpha_2 = -\frac{b}{2}}, \qquad \text{where} \quad s, t = +, -. \tag{52}
$$

But this already guarantees that the finite difference equations that follow from the Moore-Seiberg pentagon equation for the Racah coefficients as sketched in Section 3 will be exactly the same as those satisfied by the fusion coefficients $G_{\alpha_{21},\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$.

One can furthermore show that the functional equations that describe the relation between Racah coefficients with argument α_i and $Q-\alpha_i$, $i=1,\ldots,4$ coincide with those satisfied by $G_{\alpha_{21},\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$.

We conclude that the Racah coefficients solve the full system of functional equations for the fusion coefficients. This implies according to Section 3 that one has at least for real, irrational values of b

$$G_{\alpha_{21}\alpha_{32}}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]=\left\{\begin{smallmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{smallmatrix}\middle|\begin{smallmatrix}\alpha_{21}\\ \alpha_{32}\end{smallmatrix}\right\}_b. \tag{53}$$

8. DISCUSSION

The results of the present paper amount to a verification of consistency of the bootstrap for Liouville theory under the assumptions on existence and properties of fusion coefficients discussed in Sections 2 and 3.

It should be emphasized that characterizing the fusion coeffcients as Racah-coefficients for a quantum group will be important beyond the task of veryifying the consistency of previous results on three point function and spectrum of Liouville theory. It also allows one to complete the bootstrap for the associated boundary problem (Liouville theory on the strip or half-plane) that was begun in [28] and should have important applications to D-brane physics on the associated noncompact backgrounds. Specifically, it was observed by one of us (J.T.) more than two years ago that the three point function for *boundary* operators can be expressed in terms of the fusion coefficients, which was one of the main motivations for undertaking the present investigation.

In fact, due to the close relationship between Liouville theory and the H_3^+ or $SL(2)/U(1)$ WZNW models it should now not be too difficult to generalize our methods to obtain similar results for these models. For example, an exact investigation of effective field theories on D-branes in ADS_3 similar to what was done in [29, 30] is now within reach.

Our results may also be interesting from the mathematical point of view. The list of examples for non-compact quantum groups where results on the harmonic analysis are known is rather short [31, 32, 33]. In fact, the "deformation" of classical groups to quantum groups often meets subtle obstacles [25]. Here we have found an example which is *not* a deformation of a classical group but in some respects looks particularly nice (self-duality).

We would finally like to mention that there seem to be rather interesting connections of the present work to other approaches to Liouville theory, namely the Liouville model on the lattice [34] and quantization of Teichmüller space [35, 36]. In fact, in all cases the special function $S_b(x)$ (called quantum dilogarithm there) as well as the duality $b\to b^{-1}$ play crucial roles. Making contact with the quantization of Teichmüller space [35, 36] will require diagonalization of finite difference operators of similar form as have appeared in the present work [37].

9. APPENDIX: SPECIAL FUNCTIONS

The basic building block for the class of special functions to be considered is the the Double Gamma function introduced by Barnes [22], see also [38]. The Double Gamma function is defined as

$$\log\Gamma_2(s|\omega_1,\omega_2)=\left(\frac{\partial}{\partial t}\sum_{n_1,n_2=0}^{\infty}(s+n_1\omega_1+n_2\omega_2)^{-t}\right)_{t=0}. \tag{54}$$

Let $\Gamma_b(x) = \Gamma_2(x|b,b^{-1})$, and define the Double Sine function $S_b(x)$ and the Upsilon function $\Upsilon_b(x)$ respectively by

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)} \qquad \Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}. \tag{55}$$

It will also be useful to introduce

$$G_b(x) = e^{\frac{\pi i}{2}x(x-Q)} S_b(x). \tag{56}$$

The b-hypergeometric function will be defined by an integral representation that resembles the Barnes integral for the ordinary hypergeometric function:

$$F_b(\alpha,\beta;\gamma;y) = \frac{1}{i}\frac{S_b(\gamma)}{S_b(\alpha)S_b(\beta)}\int_{-i\infty}^{i\infty} ds\; e^{2\pi i s y}\frac{S_b(\alpha+s)S_b(\beta+s)}{S_b(\gamma+s)S_b(Q+s)}, \tag{57}$$

where the contour is to the right of the poles at $s=-\alpha-nb-mb^{-1}$ $s=-\beta-nb-mb^{-1}$ and to the left of the poles at $s=nb+mb^{-1}$ $s=Q-\gamma+nb+mb^{-1}$, $n,m=0,1,2,\dots$. The function $F_b(\alpha,\beta;\gamma;-ix)$ is a solution of the q-hypergeometric difference equation

$$\big([\delta_x+\alpha][\delta_x+\beta]-e^{-2\pi b x}[\delta_x][\delta_x+\gamma-Q]\big)F_b(\alpha,\beta;\gamma;-ix)=0, \qquad \delta_x=\tfrac{1}{2\pi}\partial_x \tag{58}$$

This definition of a q-hypergeometric function is closely related to the one first given in [39]. There is also a kind of deformed Euler-integral for the hypergeometric function [39]:

$$\Psi_b(\alpha,\beta;\gamma;x) = \frac{1}{i}\int_{-i\infty}^{i\infty} ds\; e^{2\pi i s\beta}\frac{G_b(s+x)G_b(s+\gamma-\beta)}{G_b(s+x+\alpha)G_b(s+Q)} \tag{59}$$

The precise relation between Ψ_b and F_b is

$$\Psi_b(\alpha,\beta;\gamma;x) = \frac{G_b(\beta)G_b(\gamma-\beta)}{G_b(\gamma)}F_b(\alpha,\beta;\gamma;x'), \qquad x' = x-\tfrac{1}{2}(\gamma-\alpha-\beta+Q). \tag{60}$$

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